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Some Exact Solutions of String Cosmological Models in Bianchi Type-II Space-Time

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Abstract The Bianchi type-II cosmological solutions of massive strings have been investigated in the presence as well as absence of the magnetic field. The energy conditions for a cloud of strings coupled to the Einstein equations have been examined. The physical features of the models have also been discussed.

Keywords Massive string · Magnetic field · Bianchi type-V model

1 Introduction

The spatially homogeneous and anisotropic cosmological models have a significant role in the description of the universe in the early stages of its evolution. The study of the cosmic strings has received considerable attention in cosmology since they play an important role in structure formation and evolution of the universe [1, 2]. The gravitational effects of cosmic strings have been extensively studied by Vilenkin [2], Goetz [3], Letelier [4], Stachel [5] in general relativity. There is no direct evidence of strings in the present day universe. Accordingly cosmological models of the universe that evolve from a string-dominated era and end up in a particle-dominated era are of physical interest. Matraverse [6] has presented a class of exact solutions of Einstein field equations with a two-parameter family of classical strings as the source of the gravitational field. Krori et al. [7] have obtained some exact solutions in string cosmology for homogeneous spaces of Bianchi types-II, VI_0, VIII, and IX. Banerjee et al. [8] have studied Bianchi type-I strings cosmological models with and without a source-free magnetic field. Tikekar and Patel [9] have obtained some exact Bianchi type-III cosmological solutions of massive strings in the presence of magnetic field. Shri Ram and Singh [10] have obtained some new exact solutions of string cosmology with and without a source-free magnetic field in the context of Bianchi type-I space-times in the different basic form. Singh and Shri Ram [11] have presented a technique to generate new exact Bianchi type-III cosmological solutions of massive strings in the presence and absence of

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the magnetic field. Singh [12] has investigated the Bianchi type-V cosmological solutions of massive strings in the presence and absence of the magnetic field. Singh [13] has discussed the new models of a string cloud with and without a source-free magnetic field in the context of Bianchi type-V space-times in normal gauge for Lyra's geometry. Cosmic anisotropy may be attributed to the large scale magnetic fields. Katore and Rane [14], Tripathy et al. [15], Singh [16], and Pradhan et al. [17] have also studied string cosmological models in the presence of the electromagnetic field.

In this paper we obtain some new exact solutions of the string cosmology in the presence and absence of the magnetic field in the context of Bianchi type-II space-times. We examine the energy conditions for a cloud of strings coupled to the Einstein equations.

2 Einstein Field Equations

We assume the metric for Bianchi type-II space-time in the general form:

$$ds^2 = dt^2 - A^2 (dx - z\,dy)^2 - B^2\,dy^2 - C^2\,dz^2, \tag{1}$$

where the metric potentials A, B, and C are functions of cosmological time t.

The Einstein equations for a cloud of massive strings are:

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij}. \tag{2}$$

The energy-momentum tensor T_{ij} for a cloud of string dust with a magnetic field along the direction of the string is given by

$$T_{ij} = \rho u_i u_j - \lambda w_i w_j + \frac{1}{4\pi}\left(-g^{\alpha\beta} F_{i\alpha} F_{j\beta} + \frac{1}{4} g_{ij} F_{\alpha\beta} F^{\alpha\beta}\right), \tag{3}$$

where ρ is the proper energy density for a cloud of strings with particles attached to them, λ denotes the string tension density, the unit time-like vector u^i describe the cloud four-velocity and the unit space-like vector w^i denotes the direction of the string which can be taken along any one of the three directions $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$. Without loss of generality let us choose y-direction as the direction of the string along which the magnetic field is assumed to be present. So that

$$w^i = \left(0, \frac{1}{\sqrt{A^2 z^2 + B^2}}, 0, 0\right).$$

In a co-moving coordinate system, we have $u^i = (0, 0, 0, 1)$. Thus we have $u_i u^i = -w^i w_j = 1$, and $u^i w_i = 0$.

The electromagnetic field tensor F_{ij} has only one non-zero component F_{31} because the magnetic field is assumed to be along the y-direction. Subsequently Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0, \qquad F^{ij}_{;j}, \tag{4}$$

lead to

$$F^{31} = K, \quad \text{where } K \text{ is a constant}. \tag{5}$$

The field equations (2) for the metric (1) lead to the following system of equations:

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}-\frac{3}{4}\frac{A^2}{B^2C^2}=2C^2A^2K^2-\frac{(A^2z^2+B^2)C^2A^2K^2}{B^2}, \tag{6}$$

$$\frac{\ddot{C}}{C}+\frac{\ddot{A}}{A}+\frac{\dot{C}\dot{A}}{CA}+\frac{1}{4}\frac{A^2}{B^2C^2}=-8\pi\lambda-C^2A^2K^2, \tag{7}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{1}{4}\frac{A^2}{B^2C^2}=\frac{(A^2z^2+B^2)C^2A^2K^2}{B^2}, \tag{8}$$

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}-\frac{1}{4}\frac{A^2}{B^2C^2}=-8\pi\rho-\frac{(A^2z^2+B^2)C^2A^2K^2}{B^2}, \tag{9}$$

where a dot denotes differentiation with respect to t. Let ρ_p denotes the particle energy density of the configuration so that

$$\rho=\rho_p+\lambda. \tag{10}$$

The energy conditions lead to $\rho\geq 0$ and $\rho_p\geq 0$, leaving sign of λ unrestricted.

3 Exact Solutions

The field equations (6)–(9) constitute a system of four equations with five unknown parameters A, B, C, ρ, and λ. Therefore some additional constraint equations relating these parameters are required to obtain explicit solutions of the system of the equations of the system. Let us assume that ρ and λ are related by Takabayasi's equation of state, $\rho=(1+W)\lambda$, $W>0$, where W is a constant. A linear combination of the system of equations (6)–(9) provides

$$(1+3W)\frac{\ddot{A}}{A}+2W\frac{\ddot{B}}{B}+3(1+W)\frac{\ddot{C}}{C}+(W-3)\frac{\dot{A}\dot{B}}{AB}+(W-1)\frac{\dot{B}\dot{C}}{BC}+2W\frac{\dot{C}\dot{A}}{CA}=0. \tag{11}$$

We assume $B=A^m$ and $C=A^n$, where m and n are arbitrary constants. Equation (11) reduces to

$$\frac{\ddot{A}}{A}+\psi\frac{\dot{A}}{A}=0, \tag{12}$$

where

$$\psi=\frac{(3n^2-3m-3n-mn)+W(2m^2+3n^2+mn-m-n)}{(3n+1)+W(2m+3n+3)}.$$

Solving (12), we get

$$A=[(\psi+1)(k_1t-k_2)]^{1/(\psi+1)}, \tag{13}$$

where k_1 and k_2 are arbitrary constants.

The physical and kinematical parameters [18] for this model have the following expressions:

$$\begin{aligned}8\pi\rho&=8\pi(1+W)\lambda\\&=(1+W)\left[\frac{k_1^2\{\psi(n+1)-n^2\}}{\{(\psi+1)(k_1t-k_2)\}^2}-\frac{1}{4\{(\psi+1)(k_1t-k_2)\}^{2(m+n-1)/(\psi+1)}}\right.\end{aligned}$$

$$- K^2\{(\psi+1)(k_1t-k_2)\}^{2(n+1)/(\psi+1)}\Bigg], \tag{14}$$

$$\rho_p = \frac{W}{8\pi}\left[\frac{k_1^2\{\psi(n+1)-n^2\}}{\{(\psi+1)(k_1t-k_2)\}^2} - \frac{1}{4\{(\psi+1)(k_1t-k_2)\}^{2(m+n-1)/(\psi+1)}}\right.$$
$$\left. - K^2\{(\psi+1)(k_1t-k_2)\}^{2(n+1)/(\psi+1)}\right], \tag{15}$$

Spatial volume:

$$V^3 = [(\psi+1)(k_1t-k_2)]^{(m+n+1)/(\psi+1)}, \tag{16}$$

Expansion scalar:

$$\Theta = \frac{k_1(m+n+1)}{(\psi+1)(k_1t-k_2)}, \tag{17}$$

Shear scalar:

$$\sigma^2 = \frac{1}{2}\left[\frac{(11n^2-m^2-14mn-10m+2n+15)k_1^2}{12\{(\psi+1)(k_1t-k_2)\}^2}\right.$$
$$+ \frac{4(m+n+1)z^2k_1^2}{3\{(\psi+1)(k_1t-k_2)\}^{(\psi+2m+1)/(\psi+1)}}[1-k_1\{(\psi+1)(k_1t-k_2)\}^{(1-\psi)/(1+\psi)}]$$
$$+ \frac{2}{3}z^4k_1^2\{(\psi+1)(k_1t-k_2)\}^{2(1-\psi)/(1+\psi)}$$
$$\left.\times\left\{2(m+n+1) - \frac{(m+n+1)^2+9}{3\{(\psi+1)(k_1t-k_2)\}^{4m/(\psi+1)}}\right\}\right], \tag{18}$$

Hubble parameter:

$$H = \frac{k_1(m+n+1)}{3(\psi+1)(k_1t-k_2)}, \tag{19}$$

Deceleration parameter:

$$q = \frac{(3\psi-m-n+2)}{(m+n+1)}, \tag{20}$$

$$\frac{\sigma^2}{\Theta} = \frac{(\psi+1)(k_1t-k_2)}{2k_1(m+n+1)}\left[\frac{(11n^2-m^2-14mn-10m+2n+15)k_1^2}{12\{(\psi+1)(k_1t-k_2)\}^2}\right.$$
$$+ \frac{4(m+n+1)z^2k_1^2}{3\{(\psi+1)(k_1t-k_2)\}^{(\psi+2m+1)/(\psi+1)}}[1-k_1\{(\psi+1)(k_1t-k_2)\}^{(1-\psi)/(1+\psi)}]$$
$$+ \frac{2}{3}z^4k_1^2\{(\psi+1)(k_1t-k_2)\}^{2(1-\psi)/(1+\psi)}\left\{2(m+n+1)\right.$$
$$\left.\left. - \frac{(m+n+1)^2+9}{3\{(\psi+1)(k_1t-k_2)\}^{4m/(\psi+1)}}\right\}\right]. \tag{21}$$

It should be noted that the universe exhibits initial singularity of the POINT-type at $t = (k_2/k_1)$. The space-time is well behaved in the range $(k_2/k_1) < t < \infty$. For physical realistic models, we take $k_1, k_2 > 0$. At the initial moment $t = (k_2/k_1)$, the physical and kinematical parameters ρ, λ, ρ_p, Θ, σ^2, and H tend to infinity and the magnetic field disappeared. So the universe begins from initial singularity with infinite energy density, infinite string tension density, infinite particle energy density, and with infinite rate of shear and expansion. Moreover ρ, λ, ρ_p, Θ, σ^2, and H tend to a finite limit as $t \to 0$. Thus ρ, λ, ρ_p, Θ, σ^2, and H are monotonically decreasing for t in the range $(k_2/k_1) < t < \infty$. Near the singularity, the particle will 'dominate' the string ($\rho_p > \lambda$). The expressions (14)–(15) indicate that the magnetic field is linked with λ, ρ, and ρ_p. The model gives solution for p-string in presence of the magnetic field. In case the model represents a geometric string model when

$$4k_1^2\{\psi(n+1) - n^2\} = \{(\psi+1)(k_1t - k_2)\}^{2(\psi-m-n+2)/(\psi+1)} + 4K^2\{(\psi+1)(k_1t-k_2)\}^{2(\psi+n+2)/(\psi+1)}.$$

We also discuss the following properties:

- Near the singularity the string tension density $\lambda \cong 0$ and $\rho \cong \rho_p$ when $\psi(n+1) \cong n^2$. At this epoch, the model can be supposed to represent a dust filled universe with magnetic field.
- When $\psi(n+1) \leq n^2$, the string phase of the universe can disappear in the interval $(k_2/k_1) < t < \infty$ because λ may become negative. In this interval, the model can be supposed to contain a highly anisotropic fluid of particles.

The energy conditions $\rho \geq 0$ and $\rho_p \geq 0$ are satisfied when

$$4k_1^2\{\psi(n+1) - n^2\} \geq \{(\psi+1)(k_1t - k_2)\}^{2(\psi-m-n+2)/(\psi+1)} + 4K^2\{(\psi+1)(k_1t-k_2)\}^{2(\psi+n+2)/(\psi+1)}.$$

The spatial volume V tends to zero at the initial singularity. As time proceeds the universe approaches toward an infinite volume in the limit as $t \to \infty$. For $m, n > 0$, the expansion scalar Θ is positive. Therefore the model describes an expanding model in the presence of magnetic field. The deceleration parameter q is constant but the ratio (σ^2/Θ) of the model is decreasing from a very large quantity to zero in the interval $(k_2/k_1) < t < \infty$. Also in limiting case $(\sigma^2/\rho) \neq 0$ as $t \to \infty$. This shows that the model is highly anisotropic at the time of the evolution of the universe.

In this model particle horizon exists because

$$\int_{t_0}^{t} \frac{dt'}{V(t')} = \left[\frac{3\{(\psi+1)(k_1t' - k_2)\}^{(3\psi-m-n+2)/3(\psi+1)}}{k_1(3\psi - m - n + 2)}\right]_{t_0}^{t}, \tag{22}$$

is a convergent integral. However the model does not admit rotation and acceleration.

4 Exact Solutions in the Absence of a Magnetic Field

The field equations (6)–(9) in the absence of magnetic field (i.e., $K = 0$) reduce to

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{3}{4}\frac{A^2}{B^2C^2} = 0, \tag{23}$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} + \frac{1}{4}\frac{A^2}{B^2C^2} = -8\pi\lambda, \tag{24}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{1}{4}\frac{A^2}{B^2C^2} = 0, \tag{25}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \frac{1}{4}\frac{A^2}{B^2C^2} = -8\pi\rho. \tag{26}$$

To obtain explicit solutions of the system of field equations (23)–(26), we assume $B = A^m$, and $C = A^n$ where m and n are arbitrary constants. The solutions are given by

$$A = [(Q+1)(\ell_1 t - \ell_2)]^{1/(Q+1)}, \quad \text{where } Q = \frac{(4m^2 + n^2 + mn - m - n)}{(4m + n + 3)} \tag{27}$$

where ℓ_1 and ℓ_2 are arbitrary constants.

The physical and kinematical parameters for this model are given by

$$8\pi\lambda = \left[\frac{\ell_1^2\{Q(n+1) - n^2\}}{\{(Q+1)(\ell_1 t - \ell_2)\}^2} - \frac{1}{4\{(Q+1)(\ell_1 t - \ell_2)\}^{2(m+n-1)/(Q+1)}}\right], \tag{28}$$

$$8\pi\rho = \left[\frac{1}{4\{(Q+1)(\ell_1 t - \ell_2)\}^{2(m+n-1)/(Q+1)}} - \frac{\ell_1^2(m + mn + n)}{\{(Q+1)(\ell_1 t - \ell_2)\}^2}\right], \tag{29}$$

$$8\pi\rho_p = \left[\frac{\ell_1^2\{n^2 - (n+1)(m+Q) - n\}}{\{(Q+1)(\ell_1 t - \ell_2)\}^2} + \frac{1}{2\{(Q+1)(\ell_1 t - \ell_2)\}^{2(m+n-1)/(Q+1)}}\right], \tag{30}$$

$$V^3 = [(Q+1)(\ell_1 t - \ell_2)]^{(m+n+1)/(Q+1)}, \tag{31}$$

$$\Theta = \frac{\ell_1(m+n+1)}{(Q+1)(\ell_1 t - \ell_2)}, \tag{32}$$

$$\begin{aligned}\sigma^2 = \frac{1}{2}\Bigg[&\frac{\ell_1^2(11n^2 - m^2 - 14mn - 10m + 2n + 15)}{12\{(Q+1)(\ell_1 t - \ell_2)\}^2} \\ &+ \frac{4(m+n+1)z^2\ell_1}{9\{(Q+1)(\ell_1 t - \ell_2)\}^{(Q+2m+1)/(Q+1)}} \\ &\times \left\{1 - \frac{\ell_1}{\{(Q+1)(\ell_1 t - \ell_2)\}^{(Q-1)/(Q+1)}}\right\} + \frac{2\ell_1^2 z^4}{3\{(Q+1)(\ell_1 t - \ell_2)\}^{2(Q-1)/(Q+1)}} \\ &\times \left\{2(m+n+1) - \frac{(m+n+1)^2 + 9}{3\{(Q+1)(\ell_1 t - \ell_2)\}^{4m/(Q+1)}}\right\}\Bigg],\end{aligned} \tag{33}$$

$$H = \frac{(m+n+1)\ell_1}{3(Q+1)(\ell_1 t - \ell_2)}, \tag{34}$$

$$q = \frac{(3Q - m - n + 2)}{(m+n+1)}. \tag{35}$$

The universe exhibits initial singularity of the POINT-type at $t = (\ell_2/\ell_1)$. For physical realistic models, we take $\ell_1, \ell_2 > 0$. The space-time is well behaved in the range $(\ell_2/\ell_1) < t < \infty$. At the initial singularity $t = (\ell_2/\ell_1)$, the physical and kinematical parameters ρ, λ, ρ_p, Θ, σ^2, and H tend to infinity. So the universe starts from initial singularity with infinite energy density, infinite string tension density, infinite particle energy density, and with infinite rate of shear and expansion. Moreover ρ, λ, ρ_p, Θ, σ^2, and H tend to zero as $t \to \infty$. Thus ρ, λ, ρ_p, Θ, σ^2, and H are monotonically decreasing toward a zero for t in the interval $(\ell_2/\ell_1) < t < \infty$. Near the singularity, the particle will 'dominate' the string

$(\rho_p > \lambda)$. The model gives solution for p-string in the absence of magnetic field. In case the model represents a cloud of geometric strings when

$$2\ell_1^2\{(n+1)(m+Q)-n^2+n\} = \{(Q+1)(\ell_1 t-\ell_2)\}^{2(Q-m-n+2)/(Q+1)}.$$

We also discuss the following properties in the interval $(\ell_2/\ell_1) < t < \infty$:

- When $t \to t^* = \infty$, the cloud of string tension density λ goes to zero and $\rho = \rho_p$. Thus the model contains only a cloud of particles.
- When $t < t^*$, the string phase of the universe disappears because λ becomes negative, i.e., the model contains a highly anisotropic fluid of particles.
- The critical instant of time $t_c = t^*$ may be calculated by knowing the critical temperature given by GUTs.

The energy conditions $\rho \geq 0$ and $\rho_p \geq 0$ are satisfied when

$$2\ell_1^2\{n^2-(n+1)(m+Q)-n\} + \{(Q+1)(\ell_1 t-\ell_2)\}^{2(Q-m-n+2)/(Q+1)} \geq 0.$$

The spatial volume V tends to zero at the initial singularity. As time proceeds the universe approaches toward an infinite volume in the limit as $t \to \infty$. For $m, n > 0$, the expansion scalar Θ is positive. Therefore the model undergoes expansion till the epoch $t = \infty$. The deceleration parameter q is constant but the ratio (σ^2/Θ) of the model is decreasing from a very large quantity to zero in the interval $(\ell_2/\ell_1) < t < \infty$. The relative anisotropy (σ^2/ρ) does not tends to zero as $t \to \infty$. Thus we observe that the model is highly anisotropic at the time of the evolution of the universe in the absence of magnetic field. This model does not admit rotation and acceleration.

In this model particle horizon exists because

$$\int_{t_0}^{t} \frac{dt'}{V(t')} = \left[\frac{3\{(Q+1)(\ell_1 t'-\ell_2)\}^{(3Q-m-n+2)/3(Q+1)}}{\ell_1(3Q-m-n+2)}\right]_{t_0}^{t}, \tag{36}$$

is a convergent integral.

5 Conclusions

We have studied that both cosmological models evolve with initial singularity of the POINT-type. The universe starts from initial singularity with infinite energy density, infinite string tension density, infinite particle energy density, and with infinite rate of shear and expansion in the presence as well as absence of a magnetic field. Both models are generated by cloud of strings with particles attached to them. For some particular cases the models of a universe which evolve from a pure geometric string-dominated era on a massive string-dominated era to a particle-dominated with or without a remnant of strings with or without a source free magnetic field. These models describe expanding models with constant deceleration parameter.

The models are highly anisotropic at the time of the evolution of the universe. The integrability and the reality conditions for a cloud of massive strings coupled to the Einstein equations in the presence and absence of a magnetic field have also been examined. These string models will be useful for better understanding of the structure formation of the universe. However both models do not admit rotation and acceleration.

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